CENTRAL LIMIT THEOREMS FOR NON-INVERTIBLE MEASURE PRESERVING MAPS

MICHAEL C. MACKEY AND MARTA TYRAN-KAMIŃSKA†

Abstract. Using the Perron-Frobenius operator we establish a new functional central limit theorem result for non-invertible measure preserving maps that are not necessarily ergodic. We apply the result to asymptotically periodic transformations and give an extensive specific example using the tent map.

1. Introduction

This paper is motivated by the question "How can we produce the characteristics of a Wiener process (Brownian motion) from a semi-dynamical system?". This question is intimately connected with central limit theorems for non-invertible maps and various invariance principles. Many results on central limit theorems and invariance principles for maps have been proved, see e.g. the surveys Denker [5] and Mackey and Tyran-Kamińska [17]. These results extend back over some decades, and include the work of Boyarsky and Scarowsky [3], Gouëzel [8], Jabłoński and Malczak [12], Rousseau-Egele [25], and Wong [32] for the special case of maps of the unit interval. Martingale approximations, developed by Gordin [7], were used by Keller [13], Liverani [16], Melbourne and Nicol [19], Melbourne and Török [20], and Tyran-Kamińska [27], to give more general results.

Throughout this paper, $(Y, \mathcal{B}, \nu)$ denotes a probability measure space and $T : Y \to Y$ a non-invertible measure preserving transformation. Thus ν is invariant under T *i.e.* $\nu(T^{-1}(A)) = \nu(A)$ for all $A \in \mathcal{B}$. The transfer operator $\mathcal{P}_T : L^1(Y, \mathcal{B}, \nu) \to L^1(Y, \mathcal{B}, \nu)$, by definition, satisfies

$$\int \mathcal{P}_T f(y) g(y) \nu(dy) = \int f(y) g(T(y)) \nu(dy)$$

for all $f \in L^1(Y, \mathcal{B}, \nu)$ and $g \in L^\infty(Y, \mathcal{B}, \nu)$.

Date: April 15, 2008.

2000 *Mathematics Subject Classification.* Primary: 37A50, 60F17; Secondary: 28D05, 60F05 .

Key words and phrases. functional central limit theorem, measure preserving transformation, Perron-Frobenius operator, maximal inequality, asymptotic periodicity, tent map.

†Corresponding author.

Let $h \in L^2(Y, \mathcal{B}, \nu)$ with $\int h(y)\nu(dy) = 0$. Define the process $\{w_n(t) : t \in [0,1]\}$ by

$$w_n(t) = \frac{1}{\sqrt{n}} \sum_{j=0}^{[nt]-1} h \circ T^j \quad \text{for} \quad t \in [0,1], \ n \geq 1 \tag{1.1}$$

(the sum from 0 to -1 is set equal to 0), where $[x]$ denotes the integer part of x. For each y, $w_n(\cdot)(y)$ is an element of the Skorohod space $D[0,1]$ of all functions which are right continuous and have left-hand limits equipped with the Skorohod topology.

$$\rho_S(\psi, \widetilde{\psi}) = \inf_{s \in \mathcal{S}} \left(\sup_{t \in [0,1]} |\psi(t) - \widetilde{\psi}(s(t))| + \sup_{t \in [0,1]} |t - s(t)| \right), \quad \psi, \widetilde{\psi} \in D[0,1],$$

where $\mathcal{S}$ is the family of strictly increasing, continuous mappings s of $[0,1]$ onto itself such that $s(0) = 0$ and $s(1) = 1$ [1, Section 14].

Let $\{w(t) : t \in [0,1]\}$ be a standard Brownian motion. Throughout the paper the notation

$$w_n \to^d \sqrt{\eta} w,$$

where η is a random variable independent of the Brownian process w, denotes the weak convergence of the sequence w_n in the Skorohod space $D[0,1]$.

Our main result, which is proved using techniques similar to those in Peligrad and Utev [22] and Peligrad et al. [23], is the following:

Theorem 1. *Let T be a non-invertible measure-preserving transformation on the probability space $(Y, \mathcal{B}, \nu)$ and let $\mathcal{I}$ be the σ-algebra of all T-invariant sets. Suppose $h \in L^2(Y, \mathcal{B}, \nu)$ with $\int h(y)\nu(dy) = 0$ is such that*

$$\sum_{n=1}^{\infty} n^{-\frac{3}{2}} \left\| \sum_{k=0}^{n-1} \mathcal{P}_T^k h \right\|_2 < \infty. \tag{1.2}$$

Then

$$w_n \to^d \sqrt{\eta} w, \tag{1.3}$$

where $\eta = E_\nu(\tilde{h}^2 | \mathcal{I})$ and $\tilde{h} \in L^2(Y, \mathcal{B}, \nu)$ is such that $\mathcal{P}_T \tilde{h} = 0$ and

$$\lim_{n \to \infty} \left\| \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (h - \tilde{h}) \circ T^j \right\|_2 = 0.$$

Recall that T is *ergodic* (with respect to ν) if, for each $A \in \mathcal{B}$ with $T^{-1}(A) = A$, we have $\nu(A) \in \{0,1\}$. Thus if T is ergodic then $\mathcal{I}$ is a trivial σ-algebra, so η in (1.3) is a constant random variable. Consequently, Theorem 1 significantly generalizes Tyran-Kamińska [27, Theorem 4], where it was assumed that T is ergodic and there is $\alpha < 1/2$ such that

$$\left\| \sum_{k=0}^{n-1} \mathcal{P}_T^k h \right\|_2 = O(n^\alpha)$$

(We use the notation $b(n) = O(a(n))$ if $\limsup_{n\to\infty} b(n)/a(n) < \infty$).

Usually, in proving central limit theorems for specific examples of transformations one assumes that the transformation is mixing. For non-invertible ergodic transformations for which the transfer operator is quasi-compact on some subspace $F \subset L^2(\nu)$ with norm $|\cdot| \geq \|\cdot\|_2$, the central limit theorem and its functional version was given in Melbourne and Nicol [19]. Since quasicompactness implies exponential decay of the L^2 norm, our result applies, thus extending the results of Melbourne and Nicol [19] to the non-ergodic case. For examples of transformations in which the decay of the L^2 norm is slower than exponential and our results apply, see Tyran-Kamińska [27].

In the case of invertible transformations, non-ergodic versions of the central limit theorem and its functional generalizations were studied in Volný [28, 29, 30, 31] using martingale approximations. In a recent review by Merlevède et al. [21], the weak invariance principle was studied for stationary sequences $(X_k)_{k\in\mathbb{Z}}$ which, in particular, can be described as $X_k = X_0 \circ T^k$, where T is a measure preserving invertible transformation on a probability space and X_0 is measurable with respect to a σ-algebra $\mathcal{F}_0$ such that $\mathcal{F}_0 \subset T^{-1}(\mathcal{F}_0)$. Choosing a σ-algebra $\mathcal{F}_0$ for a specific example of invertible transformation is not an easy task and the requirement that X_0 is $\mathcal{F}_0$-measurable may sometimes be too restrictive (see [4, 16]). Sometimes, it is possible to reduce an invertible transformation to a non-invertible one (see [20, 27]). Our result in the non-invertible case extends Peligrad and Utev [22, Theorem 1.1], which is also to be found in Merlevède et al. [21, Theorem 11], where a condition introduced by Maxwell and Woodroofe [18] is assumed. In Tyran-Kamińska [27] the condition was transformed to Equation (1.2). In the proof of our result we use Theorem 4.2 in Billingsley [1] and approximation techniques which was motivated by Peligrad and Utev [22]. The corresponding maximal inequality in our non-invertible setting is stated in Proposition 1 and its proof, based on ideas of Peligrad et al. [23], is provided in Appendix A for completeness. As in Peligrad and Utev [22], the random variable η in Theorem 1 can also be obtained as a limit in L^1, which we state in Appendix B.

The outline of the paper is as follows. Following the presentation of some background material in Section 2, we turn to a proof of our main result Theorem 1 in Section 3. Section 4 introduces asymptotically periodic transformations as a specific example of a system to which Theorem 1 applies. We analyze the specific example of an asymptotically periodic family of tent maps in Section 4.4.

2. Preliminaries

The definition of the Perron-Frobenius (transfer) operator for T depends on a given σ-finite measure μ on the measure space $(Y, \mathcal{B})$ with respect to which T is nonsingular, *i.e.* $\mu(T^{-1}(A)) = 0$ for all $A \in \mathcal{B}$ with $\mu(A) = 0$. Given such a measure the *transfer operator* $P : L^1(Y, \mathcal{B}, \mu) \to L^1(Y, \mathcal{B}, \mu)$ is

defined as follows. For any $f \in L^1(Y,\mathcal{B},\mu)$, there is a unique element Pf in $L^1(Y,\mathcal{B},\mu)$ such that

$$\int_A Pf(y)\mu(dy) = \int_{T^{-1}(A)} f(y)\mu(dy) \qquad \text{for all } A \in \mathcal{B}. \tag{2.1}$$

This in turn gives rise to different operators for different underlying measures on $\mathcal{B}$. Thus if ν is invariant for T, then T is nonsingular and the transfer operator $\mathcal{P}_T : L^1(Y,\mathcal{B},\nu) \to L^1(Y,\mathcal{B},\nu)$ is well defined. Here we write $\mathcal{P}_T$ to emphasize that the underlying measure ν is invariant under T.

The Koopman operator is defined by

$$U_T f = f \circ T$$

for every measurable $f : Y \to \mathbb{R}$. In particular, U_T is also well defined for $f \in L^1(Y,\mathcal{B},\nu)$ and is an isometry of $L^1(Y,\mathcal{B},\nu)$ into $L^1(Y,\mathcal{B},\nu)$, *i.e.* $||U_T f||_1 = ||f||_1$ for all $f \in L^1(Y,\mathcal{B},\nu)$. Since the measure ν is finite, we have $L^p(Y,\mathcal{B},\nu) \subset L^1(Y,\mathcal{B},\nu)$ for $p \geq 1$. The operator $U_T : L^p(Y,\mathcal{B},\nu) \to L^p(Y,\mathcal{B},\nu)$ is also an isometry on $L^p(Y,\mathcal{B},\nu)$.

The following relations hold between the operators $U_T, \mathcal{P}_T \colon L^1(Y,\mathcal{B},\nu) \to L^1(Y,\mathcal{B},\nu)$

$$\mathcal{P}_T U_T f = f \ \text{ and } \ U_T \mathcal{P}_T f = E_\nu(f|T^{-1}(\mathcal{B})) \tag{2.2}$$

for $f \in L^1(Y,\mathcal{B},\nu)$, where $E_\nu(\cdot|T^{-1}(\mathcal{B}))\colon L^1(Y,\mathcal{B},\nu) \to L^1(Y,T^{-1}(\mathcal{B}),\nu)$ is the operator of conditional expectation. Note that if the transformation T is invertible then $U_T\mathcal{P}_T f = f$ for $f \in L^1(Y,\mathcal{B},\nu)$.

Theorem 2. *Let T be a non-invertible measure-preserving transformation on the probability space $(Y,\mathcal{B},\nu)$ and let $\mathcal{I}$ be the σ-algebra of all T-invariant sets. Suppose that $h \in L^2(Y,\mathcal{B},\nu)$ is such that $\mathcal{P}_T h = 0$. Then*

$$w_n \to^d \sqrt{\eta} w,$$

where $\eta = E_\nu(h^2|\mathcal{I})$ is a random variable independent of the Brownian motion $\{w(t) : t \in [0,1]\}$.

Proof. When T is ergodic, a direct proof based on the fact that the family

$$\{T^{-n+j}(\mathcal{B}), \frac{1}{\sqrt{n}} h \circ T^{n-j} : 1 \leq j \leq n, n \geq 1\}$$

is a martingale difference array is given in Mackey and Tyran-Kamińska [17, Appendix A] and uses the Martingale Central Limit Theorem (cf. Billingsley [2, Theorem 35.12]) together with the Birkhoff Ergodic Theorem. This can be extended to the case of non-ergodic T by using a version of the Martingale Central Limit Theorem due to Eagleson [6, Corollary p. 561]. □

Example 1. *We illustrate Theorem 2 with an example. Let $T : [0,1] \to [0,1]$ be defined by*

$$T(y) = \begin{cases} 2y, & y \in [0, \frac{1}{4}) \\ 2y - \frac{1}{2}, & y \in [\frac{1}{4}, \frac{3}{4}), \\ 2y - 1, & y \in [\frac{3}{4}, 1]. \end{cases}$$

Observe that the Lebesgue measure on $([0,1],\mathcal{B}([0,1]))$ *is invariant for* T *and that* T *is not ergodic since* $T^{-1}([0,\frac{1}{2}]) = [0,\frac{1}{2}]$ *and* $T^{-1}([\frac{1}{2},1]) = [\frac{1}{2},1]$. *The transfer operator is given by*

$$\mathcal{P}_T f(y) = \frac{1}{2} f\left(\frac{1}{2}y\right) 1_{[0,\frac{1}{2})}(y) + \frac{1}{2} f\left(\frac{1}{2}y + \frac{1}{4}\right) + \frac{1}{2} f\left(\frac{1}{2}y + \frac{1}{2}\right) 1_{[\frac{1}{2},1]}(y).$$

Consider the function

$$h(y) = \begin{cases} 1, & y \in [0,\frac{1}{4}) \\ -1, & y \in [\frac{1}{4},\frac{1}{2}), \\ -2, & y \in [\frac{1}{2},\frac{3}{4}), \\ 2, & y \in [\frac{3}{4},1]. \end{cases}$$

A straightforward calculation shows that $\mathcal{P}_T h = 0$ *and* $E_\nu(h^2|\mathcal{I}) = 1_{[0,\frac{1}{2}]} + 4\,1_{[\frac{1}{2},1]}$. *Thus Theorem 2 shows that*

$$w_n \to^d \sqrt{E_\nu(h^2|\mathcal{I})}w.$$

In particular, the one dimensional distribution of the process $\sqrt{E_\nu(h^2|\mathcal{I})}w$ *has a density equal to*

$$\frac{1}{2}\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right) + \frac{1}{2}\frac{1}{\sqrt{8\pi t}} \exp\left(-\frac{x^2}{8t}\right), \quad x \in \mathbb{R}.$$

In general, for a given h the equation $\mathcal{P}_T h = 0$ may not be satisfied. Then the idea is to write h as a sum of two functions, one of which satisfies the assumptions of Theorem 2 while the other is irrelevant for the convergence to hold. At least a part of the conclusions of Theorem 1 is given in the following

Theorem 3 (Tyran-Kamińska [27, Theorem 3]). *Let* T *be a non-invertible measure-preserving transformation on the probability space* $(Y,\mathcal{B},\nu)$. *Suppose* $h \in L^2(Y,\mathcal{B},\nu)$ *with* $\int h(y)\nu(dy) = 0$ *is such that* (1.2) *holds. Then there exists* $\tilde{h} \in L^2(Y,\mathcal{B},\nu)$ *such that* $\mathcal{P}_T\tilde{h} = 0$ *and* $\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}(h-\tilde{h})\circ T^j$ *converges to zero in* $L^2(Y,\mathcal{B},\nu)$ *as* $n \to \infty$.

We will use the following two results for subadditive sequences.

Lemma 1 (Peligrad and Utev [22, Lemma 2.8]). *Let* V_n *be a subadditive sequence of nonnegative numbers. Suppose that* $\sum_{n=1}^\infty n^{-3/2}V_n < \infty$. *Then*

$$\lim_{m\to\infty} \frac{1}{\sqrt{m}} \sum_{j=0}^\infty \frac{V_{m2^j}}{2^{j/2}} = 0.$$

Lemma 2. *Let* V_n *be a subadditive sequence of nonnegative numbers. Then for every integer* $r \geq 2$ *there exist two positive constants* C_1, C_2 *(depending on* r*) such that*

$$C_1 \sum_{j=0}^\infty \frac{V_{r^j}}{r^{j/2}} \leq \sum_{n=1}^\infty \frac{V_n}{n^{3/2}} \leq C_2 \sum_{j=0}^\infty \frac{V_{r^j}}{r^{j/2}}.$$

Proof. When $r = 2$, the lemma follows from Lemma 2.7 in Peligrad and Utev [22], the proof of which can be easily extended to the case of arbitrary $r > 2$. $\square$

3. Maximal inequality and the proof of Theorem 1

We start by first stating our key maximal inequality which is analogous to Proposition 2.3 in Peligrad and Utev [22].

Proposition 1. *Let n, q be integers such that $2^{q-1} \leq n < 2^q$. If T is a non-invertible measure-preserving transformation on the probability space $(Y, \mathcal{B}, \nu)$ and $f \in L^2(Y, \mathcal{B}, \nu)$, then*

$$\Big\| \max_{1\leq k\leq n} \Big| \sum_{j=0}^{k-1} f \circ T^j \Big| \Big\|_2 \leq \sqrt{n} \Big(3\|f - U_T \mathcal{P}_T f\|_2 + 4\sqrt{2}\Delta_q(f) \Big), \tag{3.1}$$

where

$$\Delta_q(f) = \sum_{j=0}^{q-1} 2^{-j/2} \Big\| \sum_{k=1}^{2^j} \mathcal{P}_T^k f \Big\|_2 . \tag{3.2}$$

In what follows we assume that T is a non-invertible measure-preserving transformation on the probability space $(Y, \mathcal{B}, \nu)$.

Proposition 2. *Let $h \in L^2(Y, \mathcal{B}, \nu)$. Define*

$$h_m = \frac{1}{\sqrt{m}} \sum_{j=0}^{m-1} h \circ T^j \qquad \text{and} \qquad w_{k,m}(t) = \frac{1}{\sqrt{k}} \sum_{j=0}^{[kt]-1} h_m \circ T^{mj} \tag{3.3}$$

for $m, k \in \mathbb{N}$ and $t \in [0, 1]$. If m is such that the sequence $\|\max_{1\leq l\leq k} |w_{k,m}(l/k)|\|_2$ is bounded then

$$\lim_{n\to\infty} \big\| \sup_{0\leq t\leq 1} |w_{n,1}(t) - w_{[n/m],m}(t)| \big\|_2 = 0.$$

Proof. Let $k_n = [n/m]$. We have

$$|w_{n,1}(t) - w_{k_n,m}(t)| \leq \frac{1}{\sqrt{n}} \Big| \sum_{j=m[k_n t]}^{[nt]-1} h \circ T^j \Big| + \Big(\frac{1}{\sqrt{k_n}} - \frac{\sqrt{m}}{\sqrt{n}} \Big) \Big| \sum_{j=0}^{[k_n t]-1} h_m \circ T^{mj} \Big|,$$

which leads to the estimate

$$\begin{aligned} \big\| \sup_{0\leq t\leq 1} |w_{n,1}(t) - w_{k_n,m}(t)| \big\|_2 \leq{}& \frac{3m}{\sqrt{n}} \big\| \max_{1\leq l\leq n} |h \circ T^l| \big\|_2 \\ &+ \Big(1 - \sqrt{\frac{k_n m}{n}} \Big) \big\| \max_{1\leq l\leq k_n} |w_{k_n,m}(l/k_n)| \big\|_2 . \end{aligned} \tag{3.4}$$

Since $h \in L^2(Y, \mathcal{B}, \nu)$ we have

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \big\| \max_{1\leq l\leq n} |h \circ T^l| \big\|_2 = 0.$$

Furthermore, since the sequence $\|\max_{1\le l\le k}|w_{k,m}(l/k)|\|_2$ is bounded by assumption, and $\lim_{n\to\infty}\big(1-\sqrt{k_n m/n}\big)=0$, the second term in the right-hand side of (3.4) also tends to 0. □

Proof of Theorem 1. From Theorem 3 it follows that there exists $\tilde h\in L^2(Y,\mathcal{B},\nu)$ such that $\mathcal{P}_T\tilde h=0$ and

$$\lim_{n\to\infty}\Big\|\frac{1}{\sqrt n}\sum_{j=0}^{n-1}(h-\tilde h)\circ T^j\Big\|_2=0. \tag{3.5}$$

For each $m\in\mathbb{N}$, define

$$\tilde h_m=\frac{1}{\sqrt m}\sum_{j=1}^{m-1}\tilde h\circ T^j \qquad\text{and}\qquad \tilde w_{k,m}(t)=\frac{1}{\sqrt k}\sum_{j=0}^{[kt]-1}\tilde h_m\circ T^{mj}$$

for $k\in\mathbb{N}$ and $t\in[0,1]$. We have $\mathcal{P}_{T^m}\tilde h_m=0$ for all m. Thus Theorem 2 implies

$$\tilde w_{k,m}\to^d\sqrt{E_\nu(\tilde h_m^2|\mathcal{I}_m)}\,w \tag{3.6}$$

as $k\to\infty$, where $\mathcal{I}_m$ is the σ-algebra of T^m-invariant sets. Proposition 1, applied to T^m and $\tilde h_m$, gives

$$\big\|\max_{1\le l\le k}|\tilde w_{k,m}(l/k)|\big\|_2\le 3\|\tilde h_m\|_2.$$

Therefore, by Proposition 2, we obtain

$$\lim_{n\to\infty}\big\|\sup_{0\le t\le 1}|\tilde w_{n,1}(t)-\tilde w_{[n/m],m}(t)|\big\|_2=0$$

for all $m\in\mathbb{N}$, which implies, by Theorem 4.1 in Billingsley [1], that the limit in (3.6) does not depend on m and is thus equal to $\sqrt{E_\nu(\tilde h^2|\mathcal{I})}w$.

To prove (1.3), using Theorem 4.2 in Billingsley [1] we have to show that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\big\|\sup_{0\le t\le 1}|w_n(t)-\tilde w_{[n/m],m}(t)|\big\|_2=0. \tag{3.7}$$

Let h_m and $w_{k,m}$ be defined as in (3.3). We have

$$\begin{aligned}\big\|\sup_{0\le t\le 1}|w_n(t)-\tilde w_{[n/m],m}(t)|\big\|_2&\le\big\|\sup_{0\le t\le 1}|w_n(t)-w_{[n/m],m}(t)|\big\|_2\\&\quad+\big\|\sup_{0\le t\le 1}|w_{[n/m],m}(t)-\tilde w_{[n/m],m}(t)|\big\|_2.\end{aligned} \tag{3.8}$$

Making use of Proposition 1 with T^m and h_m we obtain

$$\big\|\max_{1\le l\le k}|w_{k,m}(l/k)|\big\|_2\le 3\big\|h_m-U_{T^m}\mathcal{P}_{T^m}h_m\big\|_2+4\sqrt2\sum_{j=0}^{\infty}2^{-j/2}\Big\|\sum_{i=1}^{2^j}\mathcal{P}_{T^m}^i h_m\Big\|_2.$$

However

$$\mathcal{P}_{T^m}h_m=\frac{1}{\sqrt m}\sum_{j=0}^{m-1}\mathcal{P}_{T^m}U_{T^j}h=\frac{1}{\sqrt m}\sum_{j=1}^{m}\mathcal{P}_T^j h,$$

by (2.2), and thus

$$\sum_{j=0}^{\infty} 2^{-j/2} \Big\| \sum_{i=1}^{2^j} \mathcal{P}_{T^m}^i h_m \Big\|_2 = \frac{1}{\sqrt{m}} \sum_{j=0}^{\infty} 2^{-j/2} \Big\| \sum_{i=1}^{m2^j} \mathcal{P}_T^i h \Big\|_2, \tag{3.9}$$

and the series is convergent by Lemma 1, which implies that the sequence $\|\max_{1\le l\le k}|w_{k,m}(l/k)|\|_2$ is bounded for all m. From Proposition 2 it follows that

$$\lim_{n\to\infty} \big\| \sup_{0\le t\le 1} |w_n(t) - w_{[n/m],m}(t)| \big\|_2 = 0.$$

We next turn to estimating the second term in (3.8). We have

$$\begin{aligned}\Big\| \sup_{0\le t\le 1} |w_{k,m}(t) - \tilde{w}_{k,m}(t)| \Big\|_2 &\le \frac{1}{\sqrt{k}} \Big\| \max_{1\le l\le k} |\sum_{j=0}^{l-1} (h_m - \tilde{h}_m)\circ T^{mj}| \Big\|_2 \\ &\le 3 \big\| h_m - \tilde{h}_m - U_{T^m}\mathcal{P}_{T^m}(h_m - \tilde{h}_m) \big\|_2 \\ &\quad + 4\sqrt{2} \sum_{j=0}^{\infty} 2^{-j/2} \Big\| \sum_{i=1}^{2^j} \mathcal{P}_{T^m}^i (h_m - \tilde{h}_m) \Big\|_2\end{aligned}$$

by Proposition 1. Combining this with (3.9) and the fact that $\mathcal{P}_{T^m}\tilde{h}_m = 0$ leads to the estimate

$$\begin{aligned}\Big\| \sup_{0\le t\le 1} |w_{k,m}(t) - \tilde{w}_{k,m}(t)| \Big\|_2 &\le 3\frac{1}{\sqrt{m}} \Big\| \sum_{j=0}^{m-1} (h - \tilde{h})\circ T^j \Big\|_2 + \frac{1}{\sqrt{m}} \Big\| \sum_{j=1}^{m} \mathcal{P}_{T^j} h \Big\|_2 \\ &\quad + \frac{4\sqrt{2}}{\sqrt{m}} \sum_{j=0}^{\infty} 2^{-j/2} \Big\| \sum_{i=1}^{m2^j} \mathcal{P}_T^i h \Big\|_2,\end{aligned}$$

which completes the proof of (3.7), because all terms on the right-hand side tend to 0 as $m\to\infty$, by (3.5) and Lemma 1.

□

4. Asymptotically periodic transformations

The dynamical properties of what are now known as asymptotically periodic transformations seem to have first been studied by Ionescu Tulcea and Marinescu [10]. These transformations form a perfect example of the central limit theorem results we have discussed in earlier sections, and here we consider them in detail.

Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. Let us write $L^1(\mu) = L^1(X, \mathcal{A}, \mu)$. The elements of the set

$$D(\mu) = \{f \in L^1(\mu) : f \ge 0 \text{ and } \int f(x)\mu(dx) = 1\}$$

are called densities. Let $T : X \to X$ be a non-singular transformation and $P : L^1(\mu) \to L^1(\mu)$ be the corresponding Perron-Frobenius operator. Then (Lasota and Mackey [15]) (T, μ) is called *asymptotically periodic* if there

exists a sequence of densities $g_1, \dots, g_r$ and a sequence of bounded linear functionals $\lambda_1, \dots, \lambda_r$ such that

$$\lim_{n\to\infty} \|P^n(f - \sum_{j=1}^{r} \lambda_j(f) g_j)\|_{L^1(\mu)} = 0 \tag{4.1}$$

for all $f \in D(\mu)$. The densities g_j have disjoint supports ($g_i g_j = 0$ for $i \neq j$) and $Pg_j = g_{\alpha(j)}$, where α is a permutation of $\{1, \dots, r\}$.

If (T, μ) is asymptotically periodic and $r = 1$ in (4.1) then (T, μ) is called *asymptotically stable* or *exact* by Lasota and Mackey [15].

Observe that if (T, μ) is asymptotically periodic then

$$g_* = \frac{1}{r} \sum_{j=1}^{r} g_j$$

is an invariant density for P, i.e. $Pg_* = g_*$. The ergodic structure of asymptotically periodic transformations was studied in Inoue and Ishitani [9].

Remark 1. *Let $\mu(X) < \infty$. Recall that P is a constrictive Perron-Frobenius operator if there exists $\delta > 0$ and $\kappa < 1$ such that for every density f*

$$\limsup_{n\to\infty} \int_A P^n f(x) \mu(dx) < \kappa$$

for all $A \in \mathcal{A}$ with $\mu(A) \leq \delta$.

It is known that if P is a constrictive operator then (T, μ) is asymptotically periodic (Lasota and Mackey [15, Theorem 5.3.1], see also Komorník and Lasota [14]), and (T, μ) is ergodic if and only if the permutation $\{\alpha(1), \dots, \alpha(r)\}$ of the sequence $\{1, \dots, r\}$ is cyclical (Lasota and Mackey [15, Theorem 5.5.1]). In this case we call r the period of T.

Let (T, μ) be asymptotically periodic and let g_* be an invariant density for P. Let $Y = \mathrm{supp}(g_*) = \{x \in X : g_*(x) > 0\}$, $\mathcal{B} = \{A \cap Y : A \in \mathcal{A}\}$, and

$$\nu(A) = \int_A g_*(x) \mu(dx), \quad A \in \mathcal{A}.$$

The measure ν is a probability measure invariant under T. In what follows we write $L^p(\nu) = L^p(Y, \mathcal{B}, \nu)$ for $p = 1, 2$. The transfer operator $\mathcal{P}_T : L^1(\nu) \to L^1(\nu)$ is given by

$$g_* \mathcal{P}_T(f) = P(f g_*) \quad \text{for} \quad f \in L^1(\nu). \tag{4.2}$$

We now turn to the study of weak convergence of the sequence of processes

$$w_n(t) = \frac{1}{\sqrt{n}} \sum_{j=0}^{[nt]-1} h \circ T^j,$$

where $h \in L^2(\nu)$ with $\int h(y)\nu(dy) = 0$, by considering first the ergodic case and then the non-ergodic case.

4.1. (T,μ) **ergodic and asymptotically periodic.** Let the transformation (T,μ) be ergodic and asymptotically periodic with period r. The unique invariant density of P is given by

$$g_* = \frac{1}{r}\sum_{j=1}^{r} g_j$$

and (T^r, g_j) is exact for every $j = 1,\dots,r$. Let $Y_j = \operatorname{supp}(g_j)$ for $j = 1,\dots,r$. Note that the set $B_j = \bigcup_{n=0}^{\infty} T^{-nr}(Y_j)$ is (almost) T^r-invariant and $\nu(B_j \setminus Y_j) = 0$ for $j = 1,\dots,r$. Since the Y_j are pairwise disjoint, we have

$$E_\nu(f|\mathcal{I}_r) = \sum_{k=1}^{r} \frac{1}{\nu(Y_k)} \int_{Y_k} f(y)\nu(dy)1_{Y_k} \quad \text{for} \quad f \in L^1(\nu),$$

where $\mathcal{I}_r$ is the σ-algebra of T^r-invariant sets. However $\nu(Y_k) = 1/r$, and thus

$$E_\nu(f|\mathcal{I}_r) = r\sum_{k=1}^{r} \int_{Y_k} f(y)\nu(dy)1_{Y_k} = \sum_{k=1}^{r} \int_{Y_k} f(y)g_k(y)\mu(dy)1_{Y_k}. \tag{4.3}$$

Theorem 4. *Suppose that $h \in L^2(\nu)$ with $\int h(y)\nu(dy) = 0$ is such that*

$$\sum_{n=1}^{\infty} n^{-\frac{3}{2}} \Big\| \sum_{k=0}^{n-1} \mathcal{P}_T^{rk} h_r \Big\|_2 < \infty, \quad \textit{where} \quad h_r = \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} h \circ T^k. \tag{4.4}$$

Then

$$w_n \to^d \sigma w,$$

where w is a standard Brownian motion and $\sigma \geq 0$ is a constant.

Moreover, if $\sum_{j=1}^{\infty} \int |h_r(y)h_r(T^{rj}(y))|\nu(dy) < \infty$ then σ is given by

$$\sigma^2 = r\Big(\int_{Y_1} h_r^2(y)\nu(dy) + 2\sum_{j=1}^{\infty} \int_{Y_1} h_r(y)h_r(T^{rj}(y))\nu(dy)\Big). \tag{4.5}$$

Proof. We have $h_r \in L^2(\nu)$ and $\int_Y h_r(y)\nu(dy) = 0$. Let

$$w_{k,r}(t) = \frac{1}{\sqrt{k}} \sum_{j=0}^{[kt]-1} h_r \circ T^{rj} \quad \text{for} \quad k \in \mathbb{N},\ t \in [0,1].$$

We can apply Theorem 1 to deduce that

$$w_{k,r} \to^d \sqrt{E_\nu(\tilde{h}_r^2|\mathcal{I}_r)}\, w \quad \text{as} \quad k \to \infty,$$

where $\mathcal{I}_r$ is the σ-algebra of all T^r invariant sets and

$$E_\nu(\tilde{h}_r^2|\mathcal{I}_r) = \lim_{n\to\infty} \frac{1}{n} E_\nu\Big(\big(\sum_{j=0}^{n-1} h_r \circ T^{rj}\big)^2 |\mathcal{I}_r \Big). \tag{4.6}$$

On the other hand, we also have

$$\sum_{j=0}^{\infty} r^{-j/2}\Big\|\sum_{k=1}^{r^j}\mathcal{P}^{rk}h_r\Big\|_2 = \sum_{j=0}^{\infty} r^{-j/2}\frac{1}{\sqrt{r}}\Big\|\sum_{k=1}^{r^{j+1}}\mathcal{P}^{k}h\Big\|_2 = \sum_{j=1}^{\infty} r^{-j/2}\Big\|\sum_{k=1}^{r^j}\mathcal{P}^{k}h\Big\|_2.$$

Thus the series

$$\sum_{n=1}^{\infty} n^{-3/2}\Big\|\sum_{k=0}^{n-1}\mathcal{P}^{k}h\Big\|_2$$

is convergent by Lemma 2. From Theorem 1 we conclude that there exists $\tilde{h} \in L^2(\nu)$ such that

$$w_n \to^d \|\tilde{h}\|_2 w$$

since T is ergodic. However

$$\|\tilde{h}\|_2 = \sqrt{E_\nu(\tilde{h}_r^2|\mathcal{I}_r)},$$

by Proposition 2. Hence $E_\nu(\tilde{h}_r^2|\mathcal{I}_r)$ is a constant and from (4.3) it follows that for each $k = 1, \ldots, r$ the integral $\int_{Y_k}\tilde{h}_r^2(y)\nu(dy)$ does not depend on k. Thus

$$\sigma^2 = \|\tilde{h}\|_2^2 = r\int_{Y_1}\tilde{h}_r^2(y)\nu(dy).$$

Since ν is T^r-invariant, we have

$$\frac{1}{n}\int_{Y_k}\Big(\sum_{j=0}^{n-1}h_r(T^{rj}(y))\Big)^2\nu(dy) = \int_{Y_k}h_r^2(y)\nu(dy) + 2\frac{1}{n}\sum_{l=1}^{n-1}\sum_{j=1}^{l}\int_{Y_k}h_r(y)h_r(T^{rj}(y))\nu(dy).$$

By assumption the sequence $(\sum_{j=1}^{n}\int_{Y_k}h_r(y)h_r(T^{rj}(y))\nu(dy))_{n\geq 1}$ is convergent to $\sum_{j=1}^{\infty}\int_{Y_k}h_r(y)h_r(T^{rj}(y))\nu(dy)$ which completes the proof when combined with (4.6) and (4.3). $\square$

4.2. (T,μ) **asymptotically periodic but not necessarily ergodic.** Now let us consider (T,μ) asymptotically periodic but not ergodic, so that the permutation α is not cyclical and we can represent it as a product of permutation cycles. Thus we can rephrase the definition of asymptotic periodicity as follows.

Let there exist a sequence of densities

$$g_{1,1}, \ldots, g_{1,r_1}, \ldots, g_{l,1}, \ldots, g_{l,r_l} \tag{4.7}$$

and a sequence of bounded linear functionals $\lambda_{1,1}, \ldots, \lambda_{1,r_1}, \ldots, \lambda_{l,1}, \ldots, \lambda_{l,r_l}$ such that

$$\lim_{n\to\infty}\|P^n(f - \sum_{i=1}^{l}\sum_{j=1}^{r_i}\lambda_{i,j}(f)g_{i,j})\|_{L^1(\mu)} = 0 \quad \text{for all} \quad f \in L^1(\mu), \tag{4.8}$$

where the densities $g_{i,j}$ have mutually disjoint supports and for each i, $Pg_{i,j} = g_{i,j+1}$ for $1 \le j \le r_i - 1$, $Pg_{i,r_i} = g_{i,1}$. Then

$$g_i^* = \frac{1}{r_i} \sum_{j=1}^{r_i} g_{i,j}$$

is an invariant density for P and (T, g_i^*) is ergodic for every $i = 1, \ldots, l$. Let g_* be a convex combination of g_i^*, i.e.

$$g_* = \sum_{i=1}^{l} \alpha_i g_i^*$$

where $\alpha_i \ge 0$ and $\sum_{i=1}^{l} \alpha_i = 1$. For simplicity, assume that $\alpha_i > 0$.

Let $Y_i = \operatorname{supp}(g_i^*)$ and $Y_{i,j} = \operatorname{supp}(g_{i,j})$, $j = 1, \ldots, r_i$, $i = 1, \ldots, l$. If $\mathcal{I}$ is the σ-algebra of all T-invariant sets, then

$$E_\nu(f|\mathcal{I}) = \sum_{i=1}^{l} \frac{1}{\nu(Y_i)} \int_{Y_i} f(y)\nu(dy) 1_{Y_i} = \sum_{i=1}^{l} \int_{Y_i} f(y) g_i^*(y)\mu(dy) 1_{Y_i}.$$

Now, if $\mathcal{I}_r$ is the σ-algebra of all T^r-invariant sets with $r = \prod_{i=1}^{l} r_i$, then

$$E_\nu(f|\mathcal{I}_r) = \sum_{i=1}^{l} \frac{r_i}{\nu(Y_i)} \sum_{j=1}^{r_i} \int_{Y_{i,j}} f(y)\nu(dy) 1_{Y_{i,j}}$$

for $f \in L^1(\nu)$, which leads to

$$E_\nu(f|\mathcal{I}_r) = \sum_{i=1}^{l} \sum_{j=1}^{r_i} \int_{Y_{i,j}} f(y) g_{i,j}(y)\mu(dy) 1_{Y_{i,j}}.$$

Using similar arguments as in the proof of Theorem 4 we obtain

Theorem 5. *Suppose that $h \in L^2(\nu)$ with $\int h(y)\nu(dy) = 0$ is such that condition (4.4) holds. Then*

$$w_n \to^d \eta w,$$

where w is a standard Brownian motion and $\eta \ge 0$ is a random variable independent of w.

Moreover, if $\sum_{j=1}^{\infty} \int |h_r(y) h_r(T^{rj}(y))| \nu(dy) < \infty$ then η is given by

$$\eta = \sum_{i=1}^{l} \frac{r_i}{\nu(Y_i)} \Big(\int_{Y_{i,1}} h_r^2(y)\nu(dy) + 2 \sum_{j=1}^{\infty} \int_{Y_{i,1}} h_r(y) h_r(T^{rj}(y))\nu(dy) \Big) 1_{Y_i}.$$

Remark 2. *Observe that condition (4.4) holds if*

$$\sum_{n=1}^{\infty} \frac{\|\mathcal{P}_T^{rn} h_r\|_2}{\sqrt{n}} < \infty.$$

The operator $\mathcal{P}_T$ is a contraction on $L^\infty(\nu)$. Therefore

$$\|\mathcal{P}_T^n f\|_2 \le \|f\|_\infty^{1/2} \|\mathcal{P}_T^n f\|_1^{1/2} \quad \text{for} \quad f \in L^\infty(\nu),\ n \ge 1,$$

which allows us to easily check condition (4.4) *for specific examples of transformations* T.

It also should be noted that, by (4.2), *we have*

$$\|\mathcal{P}_T^n f\|_1 = \|P^n(fg_*)\|_{L^1(\mu)} \quad \text{for} \quad f \in L^1(\nu),\ n \geq 1.$$

4.3. **Piecewise monotonic transformations.** Let X be a totally ordered, order complete set (usually X is a compact interval in $\mathbb{R}$). Let $\mathcal{B}$ be the σ-algebra of Borel subsets of X and let μ be a probability measure on X. Recall that a function $f : X \to \mathbb{R}$ is said to be of *bounded variation* if

$$\operatorname{var}(f) = \sup \sum_{i=1}^{n} |f(x_{i-1}) - f(x_i)| < \infty,$$

where the supremum is taken over all finite ordered sequences, (x_j) with $x_j \in X$. The bounded variation norm is given by

$$||f||_{BV} = \|f\|_{L^1(\mu)} + \operatorname{var}(f)$$

and it makes $BV = \{f : X \to \mathbb{R} : \operatorname{var}(f) < \infty\}$ into a Banach space.

Let $T : V \to X$ be a continuous map, $V \subset X$ be open and dense with $\mu(V) = 1$. We call (T, μ) a *piecewise uniformly expanding map* if:

(1) There exists a countable family $\mathcal{Z}$ of closed intervals with disjoint interiors such that $V \subset \bigcup_{Z \in \mathcal{Z}} Z$ and for any $Z \in \mathcal{Z}$ the set $Z \cap (X \setminus V)$ consists exactly of the endpoints of Z.
(2) For any $Z \in \mathcal{Z}$, $T_{|Z \cap V}$ admits an extension to a homeomorphism from Z to some interval.
(3) There exists a function $g : X \to [0, \infty)$, with bounded variation, $g_{|X \setminus V} = 0$ such that the Perron-Frobenius operator $P : L^1(\mu) \to L^1(\mu)$ is of the form
$$Pf(x) = \sum_{z \in T^{-1}(x)} g(z) f(z).$$
(4) T is expanding: $\sup_{x \in V} g(x) < 1$.

The following result is due to Rychlik [26]

Theorem 6. *If* (T, μ) *is a piecewise uniformly expanding map then it satisfies* (4.8) *with* $g_{i,j} \in BV$. *Moreover, there exist constants* $C > 0$ *and* $\theta \in (0, 1)$ *such that for every function* f *of bounded variation and all* $n \geq 1$

$$\|P^{rn} f - Q(f)\|_{L^1(\mu)} \leq C\theta^n ||f||_{BV},$$

where $r = \prod_{i=1}^{l} r_i$ *and*

$$Q(f) = \sum_{i=1}^{l} \sum_{j=1}^{r_i} \int_{Y_{i,j}} f(x)\mu(dx) g_{i,j}.$$

This result and Remark 2 imply

Corollary 1. *Let (T,μ) be a piecewise uniformly expanding map and ν an invariant measure which is absolutely continuous with respect to measure μ. If h is a function of bounded variation with $E_\nu(h|\mathcal{I}) = 0$ then condition* (4.4) *holds.*

Remark 3. *AFU-maps (Uniformly expanding maps satisfying Adler's condition with a Finite image condition, which are interval maps with a finite number of indifferent fixed points) studied in Zweimüller [35], are asymptotically periodic when they have an absolutely continuous invariant probability measure. However, the decay of the L^1 norm may not be exponential. For Hölder continuous functions h one might use the results of Young [34] to obtain bounds on this norm and then apply our results.*

4.4. **Calculation of variance for the family of tent maps using Theorem 4.** Let T be the generalized tent map on $[-1,1]$ defined by

$$T_a(x) = a - 1 - a|x| \qquad \text{for} \quad x \in [-1,1]\,, \tag{4.9}$$

where $a \in (1,2]$. The Perron-Frobenius operator $P : L^1(\mu) \to L^1(\mu)$ is given by

$$Pf(x) = \frac{1}{a}\left(f\left(\psi_a^-(x)\right) + f\left(\psi_a^+(x)\right)\right) 1_{[-1,a-1]}(x), \tag{4.10}$$

where ψ_a^- and ψ_a^+ are the inverse branches of T_a

$$\psi_a^-(x) = \frac{x+1-a}{a}, \qquad \psi_a^+(x) = -\frac{x+1-a}{a} \tag{4.11}$$

and μ is the normalized Lebesgue measure on $[-1,1]$.

Ito et al. [11] have shown that the tent map Equation 4.9 is ergodic, thus possessing a unique invariant density g_a. Provatas and Mackey [24] have proved the asymptotic periodicity of (4.9) with period $r = 2^m$ for

$$2^{1/2^{m+1}} < a \le 2^{1/2^m} \qquad \text{for} \qquad m = 0, 1, 2, \cdots .$$

Thus, for example, (T,μ) has period 1 for $2^{1/2} < a \le 2$, period 2 for $2^{1/4} < a \le 2^{1/2}$, period 4 for $2^{1/8} < a \le 2^{1/4}$, etc.

Let $Y = \operatorname{supp}(g_a)$ and $\nu_a(dy) = g_a(y)\mu(dy)$. For all $1 < a \le 2$ we have $T_a(A) = A$ with $A = [T_a^2(0), T_a(0)]$ and $g_a(x) = 0$ for $x \in [-1,1] \setminus A$. If $\sqrt{2} < a \le 2$ then g_a is strictly positive in A, thus $Y = A$ in this case. For $a \le \sqrt{2}$ we have $Y \subset A$. The transfer operator $\mathcal{P}_a\colon L^1(\nu_a) \to L^1(\nu_a)$ is given by

$$\mathcal{P}_a f = \frac{P(fg_a)}{g_a} \quad \text{for} \quad f \in L^1(\nu_a),$$

where P is the Perron-Frobenius operator (4.10).

If h is a function of bounded variation on $[-1,1]$ with $\int_{-1}^1 h(y)\nu_a(dy) = 0$ and

$$w_n(t) = \frac{1}{\sqrt{n}} \sum_{j=0}^{[nt]-1} h \circ T_a^j,$$

then there exists a constant $\sigma(h) \geq 0$ such that

$$w_n \to^d \sigma(h)w,$$

where w is a standard Brownian motion. In particular, we are going to study $\sigma(h)$ for the specific example of $h = h_a$ for $a \in (1, 2]$, where

$$h_a(y) = y - \mathfrak{m}_a,\ y \in [-1, 1], \quad \text{and} \quad \mathfrak{m}_a = \int_{[-1,1]} y g_a(y)\, dy.$$

Proposition 3. *Let $m \geq 1$ and $2^{1/2^{m+1}} < a \leq 2^{1/2^m}$. Then*

$$\sigma(h_a) = \frac{\sigma(h_{a^{2^m}})a(a-1)}{\sqrt{2^m}a^{2^m}(a^{2^m}-1)} \prod_{k=0}^{m-1}(a^{2^k}-1)^2, \tag{4.12}$$

where

$$\begin{aligned} \sigma(h_{a^{2^m}})^2 &= 2\int h_{a^{2^m}}(y) f_{a^{2^m}}(y) \nu_{a^{2^m}}(dy) - \int h^2_{a^{2^m}}(y)\nu_{a^{2^m}}(dy) \\ \text{and} \quad f_{a^{2^m}} &= \sum_{n=0}^{\infty} \mathcal{P}^n_{a^{2^m}} h_{a^{2^m}}. \end{aligned} \tag{4.13}$$

In general, an explicit representation for (4.13) is not known. Hence, before turning to a proof of Proposition 3, we first give the simplest example in which $\sigma(h_{a^{2^m}})^2$ can be calculated exactly.

Example 2. *For $a = 2$ the invariant density for the transformation T_a is $g_2 = \frac{1}{2}1_{[-1,1]}$ and the transfer operator $\mathcal{P}_2\colon L^1(\nu_2) \to L^1(\nu_2)$ has the same form as P in* (4.10)

$$\mathcal{P}_2 f = \frac{1}{2}\big(f \circ \psi_2^- + f \circ \psi_2^+\big).$$

Since $\int_{-1}^1 y dy = 0$, we have $h_2(y) = y$. We also have $\mathcal{P}_2 h_2 = 0$. Thus

$$\sigma(h_2)^2 = \frac{1}{2}\int_{-1}^1 y^2 dy = 1/3$$

and Proposition 3 gives $\sigma(h_a)$ for $a = 2^{1/2^m}$, $m \geq 1$.

We now summarize some properties of the tent map [33], which will allow us to prove Proposition 3. Let $I_0 = [x^*(a), x^*(a)(1 + \frac{2}{a})]$ and $I_1 = [-x^*(a), x^*(a)]$, where $x^*(a)$ is the fixed point of T_a other than -1, i.e.

$$x^*(a) = \frac{a-1}{a+1}.$$

Define transformations $\phi_{ia} : I_i \to [-1, 1]$ by

$$\phi_{1a}(x) = -\frac{1}{x^*(a)}x \quad \text{and} \quad \phi_{0a}(x) = \frac{a}{x^*(a)}x - a - 1.$$

We have

$$\phi_{1a}^{-1}(x) = -x^*(a)x \quad \text{and} \quad \phi_{0a}^{-1}(x) = \frac{x^*(a)}{a}\big(x + a + 1\big). \tag{4.14}$$

Then for $1 < a \le \sqrt{2}$ the map $T_a^2 : I_i \to I_i$ is conjugate to $T_{a^2} : [-1,1] \to [-1,1]$

$$T_{a^2} = \phi_{ia} \circ T_a^2 \circ \phi_{ia}^{-1} \tag{4.15}$$

and the invariant density of T_a is given by

$$g_a(y) = \frac{1}{2x^*(a)} \left(a g_{a^2}(\phi_{0a}(y)) 1_{I_0}(y) + g_{a^2}(\phi_{1a}(y)) 1_{I_1}(y) \right). \tag{4.16}$$

Lemma 3. *If $a \in (1, \sqrt{2}]$ then*

$$\mathfrak{m}_a = \frac{a-1}{2a} - \frac{(a-1)x^*(a)}{2a} \mathfrak{m}_{a^2} \tag{4.17}$$

and

$$(h_a + h_a \circ T_a) \circ \phi_{0a}^{-1} = \frac{(1-a)x^*(a)}{a} h_{a^2} \tag{4.18}$$

Proof. Equation (4.17) follows from (4.16) and (4.14), while (4.18) is a direct consequence of the definition of ϕ_{0a}^{-1}, the fact that $I_0 \subset [0,1]$, and (4.17). □

Let $m \ge 1$. For $2^{1/2^{m+1}} < a \le 2^{1/2^m}$ there exist 2^m disjoint intervals in which g_a is strictly positive and they are defined by

$$Y_j^m = \Phi_{jm}^{-1}([T_{a^{2^m}}^2(0), T_{a^{2^m}}(0)]),$$

where

$$\Phi_{jm} = \phi_{i_m a^{2^{m-1}}} \circ \phi_{i_{m-1} a^{2^{m-2}}} \circ \dots \phi_{i_2 a^2} \circ \phi_{i_1 a}$$

and $j = 1 + i_1 + 2i_2 + \dots + 2^{m-1} i_m$, $i_k = 0, 1$, $k = 1, \dots, m$. We have $T_a(Y_j^m) = Y_{j+1}^m$ for $1 \le j \le 2^m - 1$ and $T_a(Y_{2^m}^m) = Y_1^m$. In particular, we have

$$Y_1^{m+1} = \phi_{0a}^{-1}(Y_1^m) \quad \text{for} \quad m \ge 0, \tag{4.19}$$

where $Y_1^0 = [T_{a^2}^2(0), T_{a^2}(0)]$.

Lemma 4. *Define*

$$h_{r,a} = \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} h_a \circ T_a^k \quad \textit{for} \quad r \ge 1,\ a \in (1,2]. \tag{4.20}$$

Let $m \ge 0$ and $r = 2^m$. If $2^{1/4r} < a \le 2^{1/2r}$ then

$$\begin{aligned} &\int_{Y_1^{m+1}} h_{2r,a}(y) h_{2r,a}(T_a^{2rn}(y)) \nu_a(dy) = \\ &\qquad \frac{(1-a)^2 x^*(a)^2}{2^2 a^2} \int_{Y_1^m} h_{r,a^2}(y) h_{r,a^2}(T_{a^2}^{rn}(y)) \nu_{a^2}(dy) \end{aligned} \tag{4.21}$$

for all $n \ge 0$.

Proof. First observe that

$$h_{2r,a} = \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} h_{2,a} \circ T_a^{2k}. \tag{4.22}$$

Let $n \geq 0$. Since $\phi_{0a}^{-1}(\phi_{0a}(y)) = y$ for $y \in [-1,1]$, a change of variables using (4.19) and (4.16) gives

$$\int_{Y_1^{m+1}} h_{2r,a}(y) h_{2r,a}(T_a^{2rn}(y)) \nu_a(dy) = \frac{1}{2} \int_{Y_1^m} h_{2r,a}(\phi_{0a}^{-1}(y)) h_{2r,a}(T_a^{2rn}(\phi_{0a}^{-1}(y))) \nu_{a^2}(dy). \tag{4.23}$$

We have $T_a^{2k} \circ \phi_{0a}^{-1} = \phi_{0a}^{-1} \circ T_{a^2}^k$ for all $k \geq 0$ by (4.15). Thus $T_a^{2rn} \circ \phi_{0a}^{-1} = \phi_{0a}^{-1} \circ T_{a^2}^{rn}$ and from (4.22) it follows that

$$h_{2r,a} \circ \phi_{0a}^{-1} = \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} h_{2,a} \circ \phi_{0a}^{-1} \circ T_{a^2}^k.$$

By Lemma 3 we obtain

$$h_{2,a} \circ \phi_{0a}^{-1} = \frac{(1-a)x^*(a)}{\sqrt{2}a} h_{a^2}.$$

Hence

$$h_{2r,a} \circ \phi_{0a}^{-1} = \frac{(1-a)x^*(a)}{\sqrt{2}a} h_{r,a^2},$$

which, when substituted into equation (4.23), completes the proof. □

Proof of Proposition 3. First, we show that if $m \geq 1$ and $2^{1/2^{m+1}} < a \leq 2^{1/2^m}$ then

$$\sigma(h_a) = \frac{\sigma(h_{a^{2^m}})}{\sqrt{2^m} a^{2^m - 1}} \prod_{k=0}^{m-1} x^*(a^{2^k})(a^{2^k} - 1). \tag{4.24}$$

Let $m \geq 1$ and $2^{1/2^{m+1}} < a \leq 2^{1/2^m}$. Since the transformation T_a is asymptotically periodic with period 2^m, Theorem 4 gives

$$\sigma(h_a)^2 = 2^m \Big(\int_{Y_1^m} h_{2^m,a}^2(y) \nu_a(dy) + 2 \sum_{j=1}^{\infty} \int_{Y_1^m} h_{2^m,a}(y) h_{2^m,a}(T_a^{2^m j}(y)) \nu_a(dy) \Big).$$

We have $a^2 \in (2^{1/2^m}, 2^{1/2^{m-1}}]$ and the transformation T_{a^2} is asymptotically periodic with period $r = 2^{m-1}$. From (4.21) with $r = 2^{m-1}$ and Theorem 4 it follows that

$$\sigma(h_a)^2 = \frac{(a-1)^2 x^*(a)^2}{2a^2} \sigma(h_{a^2})^2.$$

Thus equation (4.24) follows immediately by an induction argument on m. Finally, we have for each $k = 0, \dots, m-1$

$$x^*(a^{2^k})(a^{2^k}-1) = \frac{a^{2^k}-1}{a^{2^k}+1}(a^{2^k}-1) = \frac{(a^{2^k}-1)^3}{a^{2^{k+1}}-1}$$

and equation (4.12) holds. Since $a^{2^m} > \sqrt{2}$ the function $f_{a^{2^m}}$ is well defined and

$$\int h_{a^{2^m}}(y) f_{a^{2^m}}(y) \nu_{a^{2^m}}(dy) = \sum_{n=0}^{\infty} \int h_{a^{2^m}}(y) h_{a^{2^m}}(T^n_{a^{2^m}}(y)) \nu_{a^{2^m}}(dy),$$

which completes the proof. □

Acknowledgments

This work was supported by the Natural Sciences and Engineering Research Council (NSERC grant OGP-0036920, Canada) and the Mathematics of Information Technology and Complex Systems (MITACS Canada). This research was carried out while MCM was visiting University of Silesia, and MT-K was visiting McGill University.

Appendix A. Proof of the maximal inequality

Proof of Proposition 1. We will prove (3.1) inductively. If $n = 1$ and $q = 1$ we have

$$\|f\|_2 \le \|f - U_T \mathcal{P}_T f\|_2 + \|U_T \mathcal{P}_T f\|_2 = \|f - U_T \mathcal{P}_T f\|_2 + \Delta_1(f)$$

by the invariance of ν under T. Now assume that (3.1) holds for all $n < 2^{q-1}$. Fix n, $2^{q-1} \le n < 2^q$. By the triangle inequality

$$\max_{1\le k\le n}\left|\sum_{j=0}^{k-1} f \circ T^j\right| \le \max_{1\le k\le n}\left|\sum_{j=0}^{k-1} (f - U_T \mathcal{P}_T f) \circ T^j\right| + \max_{1\le k\le n}\left|\sum_{j=0}^{k-1} U_T \mathcal{P}_T f \circ T^j\right|. \tag{A.1}$$

We first show that

$$\left\|\max_{1\le k\le n}\left|\sum_{j=0}^{k-1} (f - U_T \mathcal{P}_T f) \circ T^j\right|\right\|_2 \le 3\sqrt{n}\|f - U_T \mathcal{P}_T f\|_2. \tag{A.2}$$

Observe that

$$\max_{1\le k\le n}\left|\sum_{j=0}^{k-1} (f - U_T \mathcal{P}_T f) \circ T^j\right| \le \left|\sum_{j=0}^{n-1} (f - U_T \mathcal{P}_T f) \circ T^j\right| + \max_{1\le k\le n}\left|\sum_{j=1}^{k} (f - U_T \mathcal{P}_T f) \circ T^{n-j}\right|.$$

Since $\mathcal{P}_T(f - U_T\mathcal{P}_T f) = 0$, we see that

$$\left\|\sum_{j=0}^{n-1}(f - U_T\mathcal{P}_T f)\circ T^j\right\|_2 = \sqrt{n}\big\|f - U_T\mathcal{P}_T f\big\|_2.$$

For every n the family $\{\sum_{j=1}^{k}(f-U_T\mathcal{P}_T f)\circ T^{n-j} : 1 \le k \le n\}$ is a martingale with respect to $\{T^{-n+k}(\mathcal{B}) : 1 \le k \le n\}$. Thus by the Doob maximal inequality

$$\begin{aligned}\left\|\max_{1\le k\le n}\left|\sum_{j=1}^{k}(f - U_T\mathcal{P}_T f)\circ T^{n-j}\right|\right\|_2 &\le 2\left\|\sum_{j=1}^{n}(f - U_T\mathcal{P}_T f)\circ T^{n-j}\right\|_2\\ &= 2\sqrt{n}\|f - U_T\mathcal{P}_T f\|_2,\end{aligned}$$

which completes the proof of (A.2).

Now consider the second term on the right hand side of (A.1). Writing $n = 2m$ or $n = 2m+1$ yields

$$\max_{1\le k\le n}\left|\sum_{j=0}^{k-1} U_T\mathcal{P}_T f\circ T^j\right| \le \max_{1\le l\le m}\left|\sum_{j=0}^{l-1} f_1\circ T^{2j}\right| + \max_{0\le l\le m}\left|U_T\mathcal{P}_T f\circ T^{2l}\right|, \tag{A.3}$$

where $f_1 = U_{T^2}\mathcal{P}_T f + U_T\mathcal{P}_T f$. To estimate the norm of the second term in the right hand side of (A.3), observe that

$$\max_{0\le l\le m}|U_T\mathcal{P}_T f\circ T^{2l}|^2 \le \sum_{l=0}^{m}|U_T\mathcal{P}_T f\circ T^{2l}|^2,$$

which leads to

$$\left\|\max_{0\le l\le m}|U_T\mathcal{P}_T f\circ T^{2l}|\right\|_2 \le \sqrt{m+1}\|\mathcal{P}_T f\|_2, \tag{A.4}$$

since ν is invariant under T. Further, since $m < 2^{q-1}$, the measure ν is invariant under T^2, and $f_1 \in L^2(Y, \mathcal{B}, \nu)$, we can use the induction hypothesis. We thus obtain

$$\left\|\max_{1\le l\le m}\left|\sum_{j=0}^{l-1} f_1\circ T^{2j}\right|\right\|_2 \le \sqrt{m}\Big(3\|f_1 - U_{T^2}\mathcal{P}_{T^2}f_1\|_2 + 4\sqrt{2}\Delta_{q-1}(f_1)\Big).$$

We have $f_1 - U_{T^2}\mathcal{P}_{T^2}f_1 = U_T\mathcal{P}_T f - U_{T^2}\mathcal{P}_{T^2}f$, by (2.2), which implies

$$\|f_1 - U_{T^2}\mathcal{P}_{T^2}f_1\|_2 \le \|\mathcal{P}_T f\|_2 + \|\mathcal{P}_{T^2}f\|_2 \le 2\|\mathcal{P}_T f\|_2,$$

since $\mathcal{P}_T$ is a contraction. We also have

$$\begin{aligned}\Delta_{q-1}(f_1) &= \sum_{j=0}^{q-2} 2^{-j/2} \Big\| \sum_{k=1}^{2^j} \mathcal{P}_{T^2}^k f_1 \Big\|_2 = \sum_{j=0}^{q-2} 2^{-j/2} \Big\| \sum_{k=1}^{2^j} \mathcal{P}_T^{2k} f_1 \Big\|_2 \\ &= \sum_{j=0}^{q-2} 2^{-j/2} \Big\| \sum_{k=1}^{2^j} \mathcal{P}_T^{2k} (U_{T^2}\mathcal{P}_T f + U_T \mathcal{P}_T f) \Big\|_2 \\ &= \sum_{j=0}^{q-2} 2^{-j/2} \Big\| \sum_{k=1}^{2^{j+1}} \mathcal{P}_T^k f \Big\|_2 = \sqrt{2}\Big(\Delta_q(f) - \|\mathcal{P}_T f\|_2\Big).\end{aligned}$$

Therefore

$$\Big\| \max_{1\le l\le m} \Big| \sum_{j=0}^{l-1} f_1 \circ T^{2j} \Big| \Big\|_2 \le \sqrt{m}\big(8\Delta_q(f) - 2\|\mathcal{P}_T f\|_2\big),$$

which combined with (A.1) through (A.4) and the fact that $\sqrt{m+1} \le \sqrt{2m} \le \sqrt{n}$ leads to

$$\begin{aligned}\Big\| \max_{1\le k\le n} \Big| \sum_{j=1}^{k} f \circ T^{n-j} \Big| \Big\|_2 &\le 3\sqrt{n}\|f - U_T\mathcal{P}_T f\|_2 + \sqrt{m+1}\|\mathcal{P}_T f\|_2 \\ &\quad + \sqrt{2m}\big(4\sqrt{2}\Delta_q(f) - \sqrt{2}\|\mathcal{P}_T f\|_2\big) \\ &\le \sqrt{n}\big(3\|f - U_T\mathcal{P}_T f\|_2 + 4\sqrt{2}\Delta_q(f)\big). \qquad \square\end{aligned}$$

Appendix B. The limiting random variable η

Finally, we give a series expansion of $E_\nu(\tilde{h}^2|\mathcal{I})$ in Theorem 1 in terms of h and iterates of T.

Proposition 4. *Suppose $h \in L^2(Y, \mathcal{B}, \nu)$ with $\int h(y)\nu(dy) = 0$ is such that*

(B.1) $$\sum_{j=0}^{\infty} 2^{-j/2} \| \sum_{k=1}^{2^j} \mathcal{P}_T^k h\|_2 < \infty.$$

Then the following limit exists in L^1

(B.2) $$\lim_{n\to\infty} \frac{E_\nu(S_n^2|\mathcal{I})}{n} = E_\nu(h^2|\mathcal{I}) + \sum_{j=0}^{\infty} \frac{E_\nu(S_{2^j} S_{2^j} \circ T^{2^j}|\mathcal{I})}{2^j},$$

where $\mathcal{I}$ is the σ-algebra of all T-invariant sets and $S_n = \sum_{j=0}^{n-1} h \circ T^j$, $n \in \mathbb{N}$.

Moreover, if $\tilde{h} \in L^2(Y, \mathcal{B}, \nu)$ is such that $\mathcal{P}_T \tilde{h} = 0$ and $\big\| \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (h - \tilde{h}) \circ T^j \big\|_2 \to 0$ as $n \to \infty$ then

(B.3) $$E_\nu(\tilde{h}^2|\mathcal{I}) = \lim_{n\to\infty} \frac{E_\nu(S_n^2|\mathcal{I})}{n}.$$

Proof. We first prove that the series in the right-hand side of (B.2) is convergent in $L^1(Y, \mathcal{B}, \nu)$. Since $\mathcal{I} \subset T^{-2^j}(\mathcal{B})$ for all j, we see that

$$E_\nu(S_{2^j} S_{2^j} \circ T^{2^j} | \mathcal{I}) = E_\nu(E_\nu(S_{2^j} S_{2^j} \circ T^{2^j} | T^{-2^j}(\mathcal{B})) | \mathcal{I}).$$

As $S_{2^j} \circ T^{2^j}$ is $T^{-2^j}(\mathcal{B})$-measurable and integrable we have

$$E_\nu(S_{2^j} S_{2^j} \circ T^{2^j} | T^{-2^j}(\mathcal{B})) = S_{2^j} \circ T^{2^j} E_\nu(S_{2^j} | T^{-2^j}(\mathcal{B})).$$

However, $E_\nu(S_{2^j} | T^{-2^j}(\mathcal{B})) = U_T^{2^j} \mathcal{P}_T^{2^j} S_{2^j}$ from (2.2). Consequently,

$$E_\nu(S_{2^j} S_{2^j} \circ T^{2^j} | \mathcal{I}) = E_\nu(S_{2^j} \sum_{k=1}^{2^j} \mathcal{P}_T^k h | \mathcal{I}). \tag{B.4}$$

Since the conditional expectation operator is a contraction in L^1, we have

$$\| E_\nu(S_{2^j} S_{2^j} \circ T^{2^j} | \mathcal{I}) \|_1 \leq \| S_{2^j} \sum_{k=1}^{2^j} \mathcal{P}_T^k h \|_1,$$

which, by the Cauchy-Schwartz inequality, leads to

$$\| E_\nu(S_{2^j} S_{2^j} \circ T^{2^j} | \mathcal{I}) \|_1 \leq \| S_{2^j} \|_2 \| \sum_{k=1}^{2^j} \mathcal{P}_T^k h \|_2.$$

Since $\|S_{2^j}\|_2 \leq \| \max_{1 \leq l \leq 2^j} |S_l| \|_2$, the sequence $\|S_{2^j}\|_2 / 2^{j/2}$ is bounded by (B.1), Lemma 2, and Proposition 1. Hence

$$\sum_{j=0}^{\infty} \frac{\|S_{2^j}\|_2 \| \sum_{k=1}^{2^j} \mathcal{P}_T^k h \|_2}{2^j} \leq C \sum_{j=0}^{\infty} \frac{\| \sum_{k=1}^{2^j} \mathcal{P}_T^k h \|_2}{2^{j/2}} < \infty,$$

which proves the convergence in L^1 of the series in (B.2).

We now prove the equality in (B.2). Since

$$S_{2^m}^2 = \left(S_{2^{m-1}} + S_{2^{m-1}} \circ T^{2^{m-1}} \right)^2 = S_{2^{m-1}}^2 + S_{2^{m-1}}^2 \circ T^{2^{m-1}} + 2 S_{2^{m-1}} S_{2^{m-1}} \circ T^{2^{m-1}},$$

we obtain

$$E_\nu(S_{2^m}^2 | \mathcal{I}) = 2 E_\nu(S_{2^{m-1}}^2 | \mathcal{I}) + 2 E_\nu(S_{2^{m-1}} S_{2^{m-1}} \circ T^{2^{m-1}} | \mathcal{I}),$$

which leads to

$$\frac{E_\nu(S_{2^m}^2 | \mathcal{I})}{2^m} = E_\nu(h^2 | \mathcal{I}) + \sum_{j=0}^{m-1} \frac{E_\nu(S_{2^j} S_{2^j} \circ T^{2^j} | \mathcal{I})}{2^j}.$$

Thus the limit in the left-hand side of (B.2) exists for the subsequence $n = 2^m$ and the equality holds. An analysis similar to that in the proof of Proposition 2.1 in Peligrad and Utev [22] shows that the whole sequence is convergent, which completes the proof of (B.2).

We now turn to the proof of (B.3). Let $\tilde{h}$ be such that $\mathcal{P}_T\tilde{h} = 0$. Define $\tilde{S}_n = \sum_{j=0}^{n-1} \tilde{h} \circ T^j$. Substituting $\tilde{h}$ into (B.1) and (B.4) gives

$$E_\nu(\tilde{h}^2|\mathcal{I}) = \lim_{n\to\infty} \frac{E_\nu(\tilde{S}_n^2|\mathcal{I})}{n}.$$

We have

$$\left\|\frac{E_\nu(\tilde{S}_n^2|\mathcal{I})}{n} - \frac{E_\nu(S_n^2|\mathcal{I})}{n}\right\|_1 \le \left\|\frac{\tilde{S}_n^2}{n} - \frac{S_n^2}{n}\right\|_1 \le \left\|\frac{\tilde{S}_n}{\sqrt{n}} - \frac{S_n}{\sqrt{n}}\right\|_2 \left\|\frac{\tilde{S}_n}{\sqrt{n}} + \frac{S_n}{\sqrt{n}}\right\|_2$$

by the Hölder inequality, which implies (B.3) when combined with the equality $\left\|\sum_{j=0}^{n-1} \tilde{h} \circ T^j\right\|_2 = \sqrt{n}\|\tilde{h}\|_2$, and the assumption $\left\|\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}(h - \tilde{h}) \circ T^j\right\|_2 \to 0$ as $n \to \infty$. □

References

[1] Billingsley, P. (1968). *Convergence of probability measures.* John Wiley & Sons Inc., New York.

[2] Billingsley, P. (1995). *Probability and measure.* Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons Inc., New York.

[3] Boyarsky, A. and Scarowsky, M. (1979). On a class of transformations which have unique absolutely continuous invariant measures. *Trans. Amer. Math. Soc.*, 255:243–262.

[4] Conze, J.-P. and Le Borgne, S. (2001). Méthode de martingales et flot géodésique sur une surface de courbure constante négative. *Ergodic Theory Dynam. Systems*, 21:421–441.

[5] Denker, M. (1989). The central limit theorem for dynamical systems. In *Dynamical systems and ergodic theory (Warsaw, 1986)*, volume 23 of *Banach Center Publ.*, pages 33–62. PWN, Warsaw.

[6] Eagleson, G. K. (1975). Martingale convergence to mixtures of infinitely divisible laws. *Ann. Probab.*, 3(3):557–562.

[7] Gordin, M. I. (1969). The central limit theorem for stationary processes. *Dokl. Akad. Nauk SSSR*, 188:739–741.

[8] Gouëzel, S. (2004). Central limit theorem and stable laws for intermittent maps. *Probab. Theory Related Fields*, 128(1):82–122.

[9] Inoue, T. and Ishitani, H. (1991). Asymptotic periodicity of densities and ergodic properties for nonsingular systems. *Hiroshima Math. J.*, 21(3):597–620.

[10] Ionescu Tulcea, C. T. and Marinescu, G. (1950). Théorie ergodique pour des classes d'opérations non complètement continues. *Ann. of Math. (2)*, 52:140–147.

[11] Ito, S., Tanaka, S., and Nakada, H. (1979). On unimodal linear transformations and chaos. I. *Tokyo J. Math.*, 2(2):221–239.

[12] Jabłoński, M. and Malczak, J. (1983). A central limit theorem for piecewise convex mappings of the unit interval. *Tôhoku Math. J. (2)*, 35(2):173–180.

[13] Keller, G. (1980). Un théorème de la limite centrale pour une classe de transformations monotones par morceaux. *C. R. Acad. Sci. Paris Sér. A-B*, 291(2):A155–A158.

[14] Komorník, J. and Lasota, A. (1987). Asymptotic decomposition of Markov operators. *Bull. Polish Acad. Sci. Math.*, 35(5-6):321–327.

[15] Lasota, A. and Mackey, M. C. (1994). *Chaos, fractals, and noise*, volume 97 of *Applied Mathematical Sciences*. Springer-Verlag, New York.

[16] Liverani, C. (1996). Central limit theorem for deterministic systems. In *International Conference on Dynamical Systems (Montevideo, 1995)*, volume 362 of *Pitman Res. Notes Math. Ser.*, pages 56–75. Longman, Harlow.

[17] Mackey, M. C. and Tyran-Kamińska, M. (2006). Deterministic Brownian motion: The effects of perturbing a dynamical system by a chaotic semi-dynamical system. *Phys. Rep.*, 422(5):167–222.

[18] Maxwell, M. and Woodroofe, M. (2000). Central limit theorems for additive functionals of Markov chains. *Ann. Probab.*, 28(2):713–724.

[19] Melbourne, I. and Nicol, M. (2004). Statistical properties of endomorphisms and compact group extensions. *J. London Math. Soc. (2)*, 70(2):427–446.

[20] Melbourne, I. and Török, A. (2002). Central limit theorems and invariance principles for time-one maps of hyperbolic flows. *Comm. Math. Phys.*, 229(1):57–71.

[21] Merlevède, F., Peligrad, M., and Utev, S. (2006). Recent advances in invariance principles for stationary sequences. *Probab. Surv.*, 3:1–36 (electronic).

[22] Peligrad, M. and Utev, S. (2005). A new maximal inequality and invariance principle for stationary sequences. *Ann. Probab.*, 33(2):798–815.

[23] Peligrad, M., Utev, S., and Wu, W. B. (2006). A maximal L_p-inequality for stationary sequences and its applications. *Proc. Amer. Math. Soc.*, in press.

[24] Provatas, N. and Mackey, M. C. (1991). Asymptotic periodicity and banded chaos. *Phys. D*, 53(2-4):295–318.

[25] Rousseau-Egele, J. (1983). Un théorème de la limite locale pour une classe de transformations dilatantes et monotones par morceaux. *Ann. Probab.*, 11(3):772–788.

[26] Rychlik, M. (1983). Bounded variation and invariant measures. *Studia Math.*, 76(1):69–80.

[27] Tyran-Kamińska, M. (2005). An invariance principle for maps with polynomial decay of correlations. *Comm. Math. Phys.*, 260(1):1–15.

[28] Volný, D. (1987a). A nonergodic version of Gordin's CLT for integrable stationary processes. *Comment. Math. Univ. Carolin.*, 28(3):413–419.

[29] Volný, D. (1987b). On the invariance principle and functional law of iterated logarithm for nonergodic processes. *Yokohama Math. J.*, 35(1-2):137–141.

[30] Volný, D. (1989). On nonergodic versions of limit theorems. *Apl. Mat.*, 34(5):351–363.

[31] Volný, D. (1993). Approximating martingales and the central limit theorem for strictly stationary processes. *Stochastic Process. Appl.*, 44(1):41–74.

[32] Wong, S. (1979). A central limit theorem for piecewise monotonic mappings of the unit interval. *Ann. Probab.*, 7(3):500–514.

[33] Yoshida, T., Mori, H., and Shigematsu, H. (1983). Analytic study of chaos of the tent map: band structures, power spectra, and critical behaviors. *J. Statist. Phys.*, 31(2):279–308.

[34] Young, L.-S. (1999). Recurrence times and rates of mixing. *Israel J. Math.*, 110:153–188.

[35] Zweimüller, R. (1998). Ergodic structure and invariant densities of non-Markovian interval maps with indifferent fixed points. *Nonlinearity*, 11(5):1263–1276.

Departments of Physiology, Physics & Mathematics and Centre for Nonlinear Dynamics, McGill University, 3655 Promenade Sir William Osler, Montreal, QC, CANADA, H3G 1Y6

E-mail address: `michael.mackey@mcgill.ca`

Institute of Mathematics, University of Silesia, ul. Bankowa 14, 40-007 Katowice, POLAND

E-mail address: `mtyran@us.edu.pl`